# The q-deformed boson realization of representations of quantum universal enveloping algebras for q a root of unity. I The case of $U_qSL(L)$



1991 J. Phys. A: Math. Gen. 24 3265

(http://iopscience.iop.org/0305-4470/24/14/015)

J. Phys. A: Math. Gen. 24 (1991) 3265-3280. Printed in the UK

# The $q$-deformed boson realization of representations of quantum universal enveloping algebras for $q$ a root of unity: (I) the case of $U_qSL(l)$*

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Received 2 October 1990

**Abstract.** The properties of $q$-deformed boson operators with non-generic $q$ ($q$ is a root of unity) are analysed by using the representation theory method and their finite-dimensional representations are thereby obtained. Based on this discussion, reducibilities and decompositions of $q$-deformed boson-realized representations of quantum universal enveloping algebra $U_qSL(l)$ are studied for non-generic cases. The explicit matrix elements of some indecomposable representations are obtained on the $q$-deformed Fock spaces. Necessary details are provided for $U_qSL(2)$ and $U_qSL(3)$. In particular, the Lusztig operator extension of $U_qSL(2)$ is discussed in an explicit form.

## 1. Introduction

The quantum group and quantum universal enveloping algebra (QUEA) [1-6] are deeply rooted in many nonlinear physics theories through the Yang-Baxter equation [7, 8]. Recently, considerable attention has been paid to the representation theory of QUEA. The standard theory of mathematics has been developed respectively for the generic case [9, 10] and the non-generic case that $q$ is a root of unity [11, 12]. Besides these, the $q$-deformed boson (oscillator) realization, a $q$-analogue of Schwinger-Jordan mapping, of QUEA was presented independently by different authors to simplify manipulations constructing representations of QUEA in [13-15], where our discussion, as a continuation of previous work [16-18] about the usual boson realization of Lie algebras, mainly involves the QUEA $U_qSL(l) = SL_q(l)$. This method of representation theory is not only easy to comprehend for physicists, but is also a powerful tool to calculate the explicit matrix elements for the representations of QUEA. Following this work, various further investigations have been carried out in [19-24].

However, except for [19] and [24], where the non-generic case is discussed to a small extent, the discussions of the $q$-deformed boson realization mentioned above only concern the generic case that $q$ is not a root of unity and there was not a systematic

* Work supported in part by the National Science Foundation of China.

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analysis for the $q$-deformed boson realization of QUEA in the non-generic case. In this and a forthcoming paper, we will systematically study the $q$-deformed boson-realized representations of QUEA when $q$ is a root of unity, since this case is very important for physics [25-27].

This paper is arranged as follows. In section 2 we discuss the representations of the $q$-deformed boson algebra, which plays a crucial role in our problem for the non-generic case. Using the central idea in section 2, we study the decomposition structure of $q$-deformed boson-realized representations of $SL_q(2)$ for the non-generic case in section 3 and then discuss the representations of the Lusztig extension $\widehat{SL}_q(2)$ of $SL_q(2)$ explicitly in section 4. In section 5, we generalize the discussion of $SL_q(2)$ to the QUEA $SL_q(l)$ and general results are obtained. Applying them to $SL_q(3)$, we discuss $q$-deformed boson-realized representations of $SL_q(3)$ in detail for $p=3$.

In this paper the symbols $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{C}$ and $\mathbb{Z}^l$ denote respectively the set of integers, non-negative integers, the complex number field and the set of lattice points: $\{(n_1, n_2, \dots, n_l) \mid n_i \in \mathbb{Z},\ i=1,2,\dots,l\}$. According to Lusztig [11], we can consider $p$ as an odd integer $\geqslant 3$ without losing generality.

## 2. Representations of $q$-deformed boson operators for $q^p = 1$

The $q$-deformed boson ($q$-B) algebra $B$ is an associative algebra generated by the boson operators $a^+$ and $a^- = a$, $\hat{N}$ and unity that satisfy

$$aa^+ - q^{-1}a^+a = q^{\hat{N}} = Q \qquad [\hat{N}, a^{\pm}] = \pm a^{\pm} \qquad q \in \mathbb{C}. \tag{1}$$

Its elements $a$, $a^+$ and $Q$ generate its subalgebra, called $q$-deformed Heisenberg-Weyl ($q$-HW) algebra. For the generic case, the representation theory of $q$-B and $q$-HW algebras has been given in [28].

Now, we consider the non-generic case. On the $q$-deformed Fock space $F$: $\{|n\rangle = a^{+n}|0\rangle \mid n \in \mathbb{Z}_+$ and $a|0\rangle = 0, Q|0\rangle = |0\rangle\}$, we obtain an infinite-dimensional representation $\rho$

$$a^+|n\rangle = |n+1\rangle \qquad a|n\rangle = [n]|n-1\rangle \qquad Q|n\rangle = q^n|n\rangle \tag{2}$$

by using the relations

$$Qa^{\pm n} = q^{\pm n}a^{\pm n}Q \qquad aa^{+n} = [n]a^{+n-1}Q + q^{-n}a^{+n}a$$

which result from (1). Here we have defined that $[f] = (q^{+} - q^{-f})/(q - q^{-1})$ for any operator $f$ or number $f$.

Although the representation (2) is irreducible for the generic case, it is reducible for the non-generic case because there exists the singular vectors $|k \cdot p\rangle$ such that $a|k \cdot p\rangle = 0$ (this is due to $[k \cdot p] = 0$) for $k \in \mathbb{Z}_+$.

*Theorem 1.* For the non-generic case, the representation (2) is indecomposable (reducible, but not completely reducible).

*Proof.* From (2), we easily observe that there exists an invariant subspace $V^{[k]}$: $\{|kp+n\rangle \mid n \in \mathbb{Z}_+\}$ defined by a singular vector $|kp\rangle$, namely, the representation is reducible. Obviously, a complementary space $\tilde{V}^{[k]}$: $\{|n\rangle \mid n = 0, 1, 2, \dots, kp-1\}$ is not invariant. Now, we need to prove that any complementary subspace for $V^{[k]}$ is also not invariant. In fact, we suppose that there is an invariant complementary space $V'$ for $V^{[k]}$ such that $F=V^{[k]}\oplus V'$. At least it must have an element with two components separately in $V^{[k]}$ and $\tilde{V}^{[k]}$, i.e. we can let this element be

$$|x\rangle=\sum_{n=0}^{kp-1} c_n|n\rangle+\sum_{n'=kp}^{\infty} b_{n'}|n'\rangle$$

where there are a $c_n\neq 0$ and a $b_{n'}\neq 0$ at least. By action of $a^+$ on $|x\rangle$, we have a non-zero vector

$$\begin{aligned}
a^{+kp}|x\rangle&=\sum_{n=0}^{kp-1} c_n|n+kp\rangle+\sum_{n'=kp}^{\infty} b_{n'}|n'+kp\rangle\\
&=\sum_{n=0}^{\infty} c_n|n+kp\rangle\in V^{[k]}
\end{aligned}$$

$$c_n=b_n \text{ for } n=kp, kp+1, kp+2, \ldots.$$

However, since $V'$ is invariant under the action of representation (2), $a^{+kp}|x\rangle\in V'$, that is to say, $V'\cap V^{[k]}\neq\{0\}$. It is impossible because of the proposal $F=V'\oplus V^{[k]}$. Therefore, the proof is ended.

Now, considering the invariant subspace chain

$$F=V^{[0]}\supset V^{[1]}\supset V^{[2]}\supset\ldots\supset V^{[k]}\supset V^{[k+1]}\ldots$$

we observe that all the subrepresentations $\rho^{[k]}$ on invariant subspaces $V^{[k]}$ are also indecomposable. Although these representations are infinite dimensional, the quotient representation $\rho^{[k,m]}$ induced by (2) on the quotient space $Q(k,m)=V^{[k]}/V^{[m]}$ $(m>k)$:

$$\{|(k,m)n\rangle=|kp+n\rangle \bmod V^{[m]}\,|\,n=0,1,2,\ldots,(m-k)p-1\}$$

is finite dimensional and its dimension is $(m-k)p$. Using (2), we write the explicit form of $\rho^{[km]}$:

$$\begin{aligned}
&a^+|(k,m)n\rangle=|(k,m)n+1\rangle \qquad n=0,1,2,\ldots,(m-k)p-2\\
&a^+|(km)n\rangle=0 \qquad \text{for } n=(m-k)p-1\\
&a|(km)n\rangle=[n]|(km)n-1\rangle\\
&Q|(km)n\rangle=q^n|(k,m)n\rangle.
\end{aligned} \tag{3}$$

Here, it is pointed out that when $m=k+1$, the representation $\rho^{[km]}$ is irreducible. For example, for $p=3$, we obtain a 3D irreducible representation

$$a^+=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\qquad a=\begin{pmatrix}0&1&0\\0&0&[2]\\0&0&0\end{pmatrix}\qquad Q=\begin{pmatrix}1&0&0\\0&q&0\\0&0&q^2\end{pmatrix} \tag{4}$$

on the quotient space $Q(k,k+1)$: $\{|(k,k+1)0\rangle, |(k,k+1)1\rangle, |(k,k+1)2\rangle\}$. It is easy to check that (4) satisfies (1) by noticing $q^3=1$.

The above discussion is naturally generalized to the case of many bosons with the operators $a_i^-=a_i$, $a_i^+$ and $\hat{N}_i$ satisfying

$$\begin{aligned}
&a_ia_j^+=\begin{cases}a_j^+a_i & \text{for } i\neq j\\ q^{-1}a_i^+a_i+q^{\hat{N}_i}\equiv q^{-1}a_i^+a_i+Q_i & \text{for } i\neq j\end{cases}\\
&[\hat{N}_i, a_j^\pm]=\delta_{ij}(\pm a_j^\pm) \qquad [\hat{N}_i,\hat{N}_j]=[a_i^\pm, a_j^\pm]=0
\end{aligned} \tag{5}$$

where $i=1,2,\ldots,l$.

Because of the indecomposable properties mentioned above, the representations of QUEA in terms of the $q$-deformed boson operators have new reducible structures.

## 3. Representations of $SL_q(2)$

The $q$-deformed boson realizations of the generators $J_\pm$ and $J_3$ for the QUEA $SL_q(2)$ are

$$J_+ = a_1^+ a_2 \qquad J_- = a_2^+ a_1 \qquad J_3 = \hat{N}_1 - \hat{N}_2. \tag{6}$$

On the two-state $q$-deformed Fock space

$$F_2: \{|n_1, n_2\rangle = a_1^{+n_1} a_2^{+n_2}|0\rangle \,|\, n_1, n_2 \in \mathbb{Z}_+, a_i|0\rangle = \hat{N}_i|0\rangle = 0, i = 1, 2\}$$

the representation of $SL_q(2)$ [14],

$$\begin{aligned} J_+|n_1, n_2\rangle &= [n_2]|n_1+1, n_2-1\rangle \\ J_-|n_1 n_2\rangle &= [n_1]|n_1-1, n_2+1\rangle \\ J_3|n_1, n_2\rangle &= (n_1 - n_2)|n_1, n_2\rangle \end{aligned} \tag{7}$$

is obtained from the realization (6). On the invariant subspace

$$V_2^{[N]}: \{f_N(n) = |n, N-n\rangle \,|\, n = 0, 1, 2, \ldots, N \in \mathbb{Z}\}$$

the above representation subduces a $(N+1)$-dimensional representation $\Gamma$:

$$\begin{aligned} J_+ f_N(n) &= [N-n] f_N(n+1) \\ J_- f_N(n) &= [n] f_N(n-1) \\ J_3 f_N(n) &= (2n - N) f_N(n) \end{aligned} \tag{8}$$

which is irreducible for the generic case.

However, for the non-generic case, there are two singular vectors $f_N(\alpha p)$ and $f_N(N - \beta p)$ such that

$$J_- f_N(\alpha p) = 0, J_+ f_N(N - \beta p) = 0 \tag{9}$$

for two positive integers $\alpha$ and $\beta \leqslant N/p$. It follows from (8) and (9) that the subspaces

$$U_\alpha = \{f_N(\alpha p + n) \,|\, n = 0, 1, 2, \ldots, N - \alpha p\}$$

and

$$W_\beta = \{f_N(N - \beta p - k) \,|\, k = 0, 1, 2, \ldots, N - \beta p\}$$

are *invariant; and* $U_{\alpha'}$ *and* $W_{\beta'}$ $(\alpha' > \alpha, \beta' > \beta)$ are respectively the invariant subspaces of $U_\alpha$ and $W_\beta$. Thus, the representation (8) and its subrepresentations on $U_\alpha$ and $W_\beta$ are reducible in the non-generic case.

According to the singular vectors $f_N(\alpha p)$ and $f_N(N - \beta p)$, there are three types of decomposition for the representation space $V_2^{[N]}$ relating to the characters of $U_\alpha \cap W_\beta$.

*Type I.* When $\alpha p - 1 > N - \beta p$, $U_\alpha \cap W_\beta = \{0\}$, the representation (8) is indecomposable. This can be proved by the same method as that for the proof of theorem 1.

*Type II.* When $\alpha p - 1 = N - \beta p$, we have $f_N(\alpha p - 1) = f_N(N - \beta p)$ and

$$\begin{aligned} J_+ f_N(\alpha p - 1) &= J_+ f_N(N - \beta p) = 0 \\ J_- f_N(\alpha p) &= 0 \end{aligned}$$

that is to say,

$$V_2^{[N]} = U_\alpha \oplus W_\beta \qquad U_\alpha \cap W_\beta = \{0\}.$$

Therefore, the representation (7) is decomposed into a direct sum of two subrepresentations separately on $U_\alpha$ and $W_\beta$, namely, the representation (8) is completely reducible.

*Type III.* When $\alpha p - 1 < N - \beta p$,

$$U_\alpha \cap W_\beta = \{f_N(\alpha p), f_N(\alpha p+1), f_N(p+2), \ldots, f_N(N-\beta p)\}$$

is a smaller invariant subspace, which does not have an invariant complementary space. Thus, the representation (7) is also indecomposable.

Now, as examples, we discuss the case of $p=3$ for $N=3$, 4, 5 and 6. In terms of the matrix units $E_{ij}$ such that

$$(E_{i,j})_{kl} = \delta_{ik}\delta_{jl}$$

we write the explicit matrices of the representations for $N=3$,

$$\begin{aligned} J_+ &= [2]E_{3,2} + E_{4,3} \\ J_- &= E_{1,2} + [2]E_{2,3} \\ J_3 &= -\tfrac{3}{2}E_{1,1} - \tfrac{1}{2}E_{2,2} + \tfrac{1}{2}E_{3,3} + \tfrac{3}{2}E_{4,4} \end{aligned} \tag{10}$$

for $N=4$,

$$\begin{aligned} J_+ &= E_{2,1} + [2]E_{4,3} + E_{5,4} \\ J_- &= E_{1,2} + [2]E_{2,3} + E_{4,5} \\ J_3 &= -2E_{1,1} - E_{2,2} + E_{4,4} + 2E_{5,5} \end{aligned} \tag{11}$$

for $N=5$,

$$\begin{aligned} J_+ &= [2]E_{2,1} + E_{3,2} + [2]E_{5,4} + E_{6,5} \\ J_- &= E_{1,2} + [2]E_{2,3} + E_{4,5} + [2]E_{5,6} \\ J_3 &= -\tfrac{5}{2}E_{1,1} - \tfrac{3}{2}E_{2,2} - \tfrac{1}{2}E_{3,3} + \tfrac{1}{2}E_{4,4} + \tfrac{3}{2}E_{5,5} + \tfrac{5}{2}E_{6,6} \end{aligned} \tag{12}$$

and for $N=6$,

$$\begin{aligned} J_+ &= [2]E_{3,2} + E_{4,3} + [2]E_{6,5} + E_{7,6} \\ J_- &= E_{1,2} + [2]E_{2,3} + [2]E_{4,5} + E_{5,6} \\ J_3 &= -3E_{1,1} - 2E_{2,2} - E_{3,3} + E_{5,5} + 2E_{6,6} + 3E_{7,7}. \end{aligned} \tag{13}$$

The decomposition of these representations is illustrated in figures 1($a$–$d$) where the upward and downward arrows denote the actions of $J_+$ and $J_-$ separately. It is easily observed from figure 1 that the representations (10) and (11) possess the reducibility of type I; the representations (12) and (13) possess reducibilities of type II and type III separately.

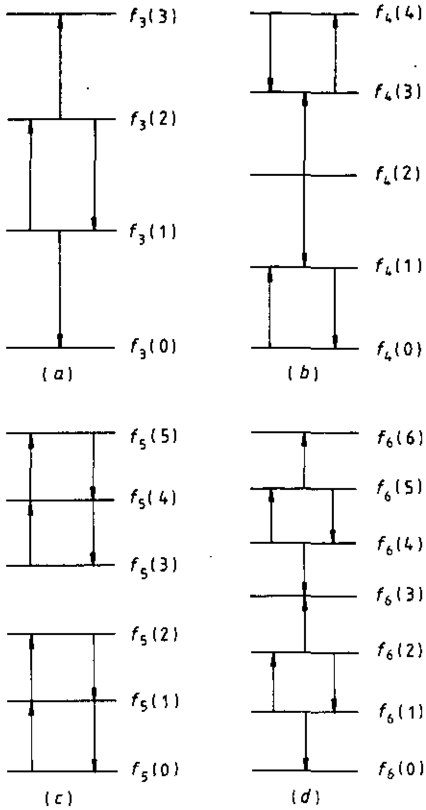


**Figure 1.** Reductions of representations for (*a*) $N=3$, (*b*) $N=4$, (*c*) $N=5$ and (*d*) $N=6$.

## 4. Lusztig operators

According to the PBW [10] for QUEA, the basis for $SL_q(2)$ can be chosen as

$$u(m, n, k) = J_+^m J_-^n J_3^k \qquad m, n, k \in \mathbb{Z}_+.$$

For any $x \in SL_q(2)$,

$$x = \sum_{m,n,k=0}^{\infty} C_{mnk} u(m, n, k)$$

where $C_{mnk}$ ($\in \mathbb{C}$) usually are not infinite. We can regard $x$ as an operator on a representation space $V$. For a given representation space $V$ of $SL_q(2)$, we extend $SL_q(2)$ to include a class of operators

$$e = \sum_{m,n,k=0}^{\infty} E_{mnk} u(m, n, k) \qquad E_{m,nk} \in \mathbb{C}$$

such that their actions on $V$ possess finite limit, where some coefficients $E_{mnk}$ must be infinite. The extended $SL_q(2)$ is denoted by $\widehat{SL}_q(2)$ and a representation of $SL_q(2)$ is still a representation of $\widehat{SL}_q(2)$, but a representation is not definitely reducible for $\widehat{SL}_q(2)$ even if it is reducible for $SL_q(2)$.

According to Lusztig [11], we introduce the Lusztig operators

$$L_\pm = \lim_{q^p \to 1} [(1/[p]!) J_\pm^p]$$

to extend $\widehat{\mathrm{SL}}_q(2)$ for the representation space $V_2^{[N]}$. We have the following theorem.

*Theorem 2.* The actions of the Lusztig operators $L_\pm$ on the space $V_2^{[N]}$ are finite and

$$L_- f_N(n) = \begin{cases} 0 & n < p \\ \alpha f_N(n-p) & n = \alpha p + n', \mathbb{Z}_+ \ni \alpha \geqslant 1, 0 \leqslant n' \leqslant p-1 \end{cases} \tag{14}$$

$$L_+ f_N(n) = \begin{cases} 0 & n > N-p \\ \beta f_N(n+p) & N-n = \beta p + m, \mathbb{Z}_+ \ni \beta \geqslant 1, 0 \leqslant m' \leqslant p-1 \end{cases} \tag{15}$$

*Proof.* Using (7) and

$$[n] = [\alpha p + n'] = [n'] \qquad \lim_{q^p \to 1} ([\alpha P]/[p]) = \alpha$$

we obtain $J_-^p f_N(n) = 0$ when $n < p$; when $n \geqslant p$,

$$\begin{aligned} J_- f_N(n) &= [\alpha p + n'][\alpha p + n' - 1][\alpha p + n' - 2] \ldots \\ &\quad \times [\alpha p + n' - p + 2][\alpha p + n' - p + 1] f_N(n-p) \\ &= [n'][n'-1][n'-2] \ldots [1][\alpha p][p-1][p-2] \ldots [n'+2][n'+1] f_N(n-p) \\ &= [\alpha p][p-1]! f_N(n-p) = 0. \end{aligned}$$

Then,

$$L_- f_N(n) = \lim_{q^p \to 1} ([\alpha p]/[p]) f_N(n-p) = \alpha f_N(n-p).$$

Using the same method, we prove (15).

Now, according to this theorem, we analyse decompositions and reducibilities of the representation (7) as a representation of $\widehat{\mathrm{SL}}_q(2)$. Because of the actions of $L_\pm$ on $f_N(n)$ such that

$$L_- f_N(\alpha p) = f_N[(\alpha - 1)p]$$

$$L_+ f_N(N - \beta p) = (\alpha' - \beta) f_N[N - (\beta - 1)p] \qquad N = \alpha' p + N', 0 \leqslant N' \leqslant p - 1$$

the subspaces $U_\alpha$ and $W_\beta$ are no longer invariant for $\widehat{\mathrm{SL}}_q(2)$. As follows, we make a concrete analysis for the reducibilities and decomposations of representations (10)–(13).

(i) In representation (10), there are two 1D $\mathrm{SL}_q(2)$-invariant subspaces, $\{f_3(0)\}$ and $\{f_3(3)\}$, but they transform into each other under the actions of $L_\pm$. Hence, only $\{f_3(0), f_3(3)\}$ is an $\widehat{\mathrm{SL}}_q(2)$-invariant subspace;

(ii) In representation (11), there two 2D $\mathrm{SL}_q(2)$-invariant subspaces, $\{f_4(0), f_4(1)\}$ and $\{f_4(3), f_4(4)\}$, but they transform into each other under the actions of $L_\pm$. Hence, their union $\{f_4(0), f_4(1), f_4(3), f_4(4)\}$ is $\widehat{\mathrm{SL}}_q(2)$ invariant.

*(iii) In* representation (12), there are two 3D $\mathrm{SL}_q(2)$-invariant subspaces, $\{f_5(0), f_5(1), f_5(2)\}$ and $\{f_5(3), f_5(4), f_5(5)\}$, and

$$V_2^{[5]} = \{f_5(0), f_5(1), f_5(2)\} \oplus \{f_5(3), f_5(4), f_5(5)\}.$$

Thus, as a representation of $\mathrm{SL}_q(2)$, (12) is completely reducible. However, due to the actions of $L_\pm$, the whole space $V_2^{[5]}$ carries an irreducible representation of $\widehat{\mathrm{SL}}_q(2)$;

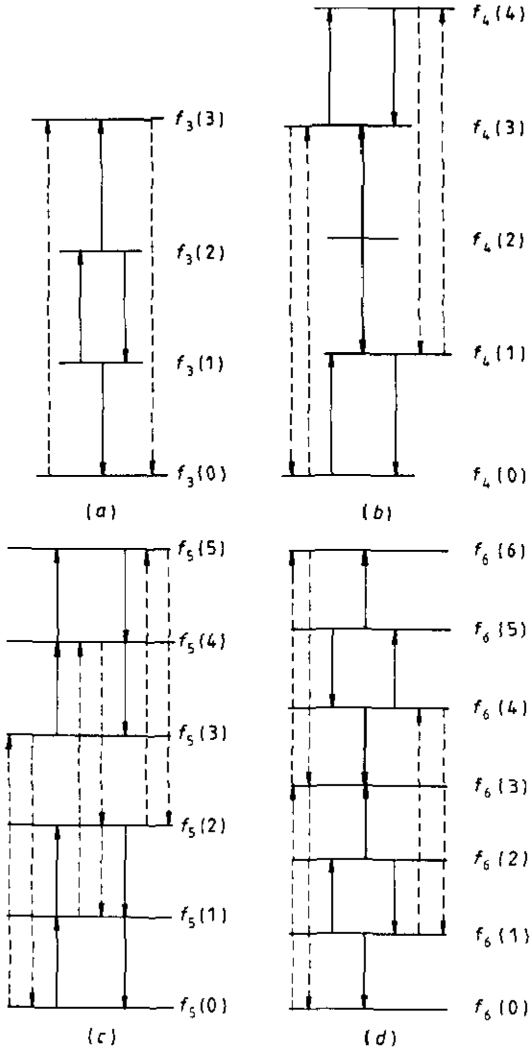


**Figure 2.** *Representations of* $\widehat{\mathrm{SL}}_q(2)$ *for* $(a)$ $N=3$, $(b)$ $N=4$, $(c)$ $N=5$ and $(d)$ $N=6$.

(iv) In representation (13), there are three 1D $\mathrm{SL}_q(2)$-invariant subspaces, $\{f_6(0)\}$, $\{f_6(3)\}$ and $\{f_6(6)\}$. They transform into one another under the actions of $L_\pm$. Hence, they span a 3D $\widehat{\mathrm{SL}}_q(2)$-invariant subspace.

The above is illustrated in figure $2(a$–$d)$ where the broken upward and downward arrows denote the actions of $L_+$ and $L_-$ separately.

## 5. Representations of $\mathrm{SL}_q(l)$: general discussion

In this and the following sections, we generalize the method for $\mathrm{SL}_q(2)$ in the last section to the general case of $\mathrm{SL}_q(l)$ when $q$ is a root of unity. As we well know, $\mathrm{SL}_q(l)$ $(l \geqslant 3)$ are associated with the standard $R$-matrices for the Yang–Baxter equation as well as $\mathrm{SL}_q(2)$ in the standard case that the usual irreducible representations are used

[4]. Recently, we obtained new $R$-matrices besides the standard ones by constructing and studying the new boson representations of $SL_q(2)$ in detail [29]. A similar situation should naturally apply to $SL_q(l)$ ($l \geq 3$). Thus, it is necessary to provide sufficient details of the new representations of $SL_q(l)$ for the construction of the new $R$-matrices associated with $SL_q(l)$ as follows.

The $q$-deformed boson realization of QUEA $SL_q(l)$ is

$$\begin{aligned} &H_i = \hat{N}_i - \hat{N}_{i+1} \\ &E_i = a_i^+ a_{i+1} \qquad F_i = a_{i+1}^+ a_i, \; i = 1, 2, \ldots, l-1. \end{aligned} \tag{16}$$

The basic relations (5) ensure that

$$\begin{aligned} &[H_i, H_j] = 0 \\ &[H_i, E_j] = \alpha_{ij} E_j \qquad [H_i, F_j] = -\alpha_{ij} F_j \\ &[E_i, F_j] = \delta_{ij}[H_j] \\ &G_j^2 G_{j\pm1} - (q + q^{-1}) G_j G_{j\pm1} G_j + G_{j\pm1} G_j^2 = 0 \end{aligned} \tag{17}$$

where $\alpha_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$ is the element of the Cartan matrix $\alpha$ of $A_{l-1}$ and $G_j = E_i$ or $F_i$.

On the $q$-deformed Fock space

$$\begin{aligned} F_l\colon \quad \{&|m\rangle = |m_1, m_2, \ldots, m_l\rangle = a_1^{+m_1} a_2^{+m_2} a_3^{+m_3} \ldots a_l^{+m_l}|0\rangle \\ &a_i|0\rangle = \hat{N}_i|0\rangle = 0, \; m_i \in \mathbb{Z}_+, \; i = 1, 2, \ldots, l\} \end{aligned}$$

we obtain a representation of $SL_q(l)$ 14

$$\begin{aligned} &H_i|\boldsymbol{m}\rangle = (m_i - m_{i+1})|\boldsymbol{m}\rangle \\ &E_i|\boldsymbol{m}\rangle = [m_{i+1}]|\boldsymbol{m} + \boldsymbol{e}_i - \boldsymbol{e}_{i+1}\rangle \\ &F_i|\boldsymbol{m}\rangle = [m_i]|\boldsymbol{m} + \boldsymbol{e}_{i+1} - \boldsymbol{e}_i\rangle \qquad i = 1, 2, \ldots, l-1 \end{aligned} \tag{18}$$

where $\boldsymbol{m} = (m_1, m_2, \ldots, m_l) \in \mathbb{Z}_+^l$ and

$$\boldsymbol{e}_1 = (1, 0, \ldots, 0), \; \boldsymbol{e}_2 = (0, 1, \ldots, 0), \ldots, \boldsymbol{e}_l = (0, 0, \ldots, 1)$$

are linear-independent unit vectors in $\mathbb{Z}^l$.

It follows from (18) that the vector $|\boldsymbol{m}\rangle$ for the representation (18) possesses a certain weight $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_{l-1}) = (m_1 - m_2, m_2 - m_3, \ldots, m_{l-1} - m_l)$ and different labels $(m_1, m_2, \ldots, m_l)$ and $(m_1 + c, m_2 + c, \ldots, m_l + c)$ ($c \in \mathbb{C}$) correspond to the same weight $\Lambda$. The latter is because the representation given by (18) is reducible. In fact, the sum $\Sigma_{i=1}^l m_i$ of the labels $m_i$ is invariant and then $V_l^{[N]}$: $\{|\boldsymbol{m}\rangle | \Sigma_{i=1}^l m_i = N\}$ for a fixed $N \in \mathbb{Z}^+$ span an invariant subspace for the representation (18). Constrained on the invariant subspace $V_l^{[N]}$, the $\boldsymbol{m}$ such that $\Sigma_{i=1}^l m_i = N$ uniquely label the state vectors and define the corresponding weight $\Lambda = (m_1 - m_2, m_2 - m_3, \ldots, m_{l-1} - m_l)$.

For convenience, in the analysis of representation reduction as follows, we introduce new labels $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_{l-1})$ where $\lambda_{i-1} = 0, 1, 2, \ldots, \lambda_i$ for a given $\lambda_i$ ($\lambda_0 = 0, \lambda_l = N$; $i = 1, 2, \ldots, l$), which are equivalent to the constrained labels $\boldsymbol{m}$. Then, we rewrite the basis

$$f_N(\boldsymbol{\lambda}) = f_N(\lambda_1, \lambda_2, \ldots, \lambda_{l-1}) = |\lambda_1 - \lambda_0, \lambda_2 - \lambda_1, \ldots, \lambda_{l-1} - \lambda_{l-2}, \lambda_l - \lambda_{l-1}\rangle$$

for the invariant subspace $V_l^{[N]}$ where $\lambda_0=0$ and $\lambda_l=N$. On the space $V_l^{[N]}$ the representation (18) defines a finite-dimensional subrepresentation

$$E_i f_N(\boldsymbol{\lambda})=[\lambda_{i+1}-\lambda_i]f_N(\boldsymbol{\lambda}+\boldsymbol{e}_i) \tag{19a}$$

$$F_i f_N(\boldsymbol{\lambda})=[\lambda_i-\lambda_{i-1}]f_N(\boldsymbol{\lambda}-\boldsymbol{e}_i) \tag{19b}$$

$$H_i f_N(\boldsymbol{\lambda})=(2\lambda_i-\lambda_{i+1}-\lambda_{i-1})f_N(\boldsymbol{\lambda}) \tag{19c}$$

whose dimension is

$$d(N,l)=\frac{(N+l-1)!}{(l-1)!N!}. \tag{20}$$

Here, $\boldsymbol{\lambda}$ is in a domain $\Delta^{l-1}$: $\{\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_{l-1})\in\mathbb{Z}^{l-1}|\lambda_0=0,\lambda_N=N,\lambda_{i-1}=0,1,2,\ldots,\lambda_i$ for a given $\lambda_i$, $i=0,1,2,\ldots,l\}$ of $\mathbb{Z}^{l-1}$ and $\boldsymbol{e}_i\in\mathbb{Z}^{l-1}$. For the generic case, (19) is irreducible and has the highest weight $\bar{\Lambda}=(N,0,0,\ldots,0)$ corresponding to the highest-weight vector $f_N(N,N,\ldots,N)=|N,0,\ldots,0\rangle$. Thus, the representation (19) is a completely symmetrized representation [14].

Now, we consider the non-generic case. Because each vector $f_N(\boldsymbol{\lambda})$ in the space $V_l^N$ corresponds to a sole lattice point $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_{l-1})\in\Delta^{l-1}\subset\mathbb{Z}^{l-1}$, we can describe the action of representation (19) on the basis $f_N(\boldsymbol{\lambda})$ by the move of the lattice point $\boldsymbol{\lambda}$. Define a hyperplane

$$\pi_i^\alpha:\quad \{\boldsymbol{\lambda}\in\mathbb{Z}_+^{l-1}|\lambda_{i+1}-\lambda_i=\alpha p\}$$

in the lattice space $\mathbb{Z}^{l-1}$. It cuts a domain $\Delta_i^\alpha$:

$$\{\boldsymbol{\lambda}\in\mathbb{Z}^{l-1}|\lambda_{i+1}-\lambda_i\geqslant\alpha p\}$$

out of $\Delta^{l-1}$. Then, we have the following theorem.

*Theorem 3.* All the vectors $f_N(\lambda)$ in $V_l^N$ corresponding to all the lattices in the domain $\Delta_i^\alpha$ span an invariant subspace $V_{\alpha i}$ of $V_l^N$ under the action of representation (19).

*Proof.* It follows from (19$a$) and (19$b$) that

$$E_{i+1}f_N(\boldsymbol{\lambda})=[\lambda_{i+2}-\lambda_{i+1}]f_N(\boldsymbol{\lambda}+\boldsymbol{e}_{i+1}) \tag{21a}$$

$$F_{i+1}f_N(\boldsymbol{\lambda})=[\lambda_{i+1}-\lambda_i]f_N(\boldsymbol{\lambda}-\boldsymbol{e}_{i+1}). \tag{21b}$$

Define the subspace $W(i,k)$: $\{f_N(\boldsymbol{\lambda})\in V_l^N|\lambda_{i+1}-\lambda_i=k\}$. Then,

$$V_{\alpha i}=\sum_{k=\alpha p}^{\infty}\oplus\ W(i,k).$$

From (21) and (19) we observe that, for $f_N(\lambda)\in W(i,k)(k\geqslant\alpha p)$,

$$E_{i+1}f_N(\boldsymbol{\lambda})\in W(i,k+1)\subset V_{\alpha i}$$

$$F_i f_N(\boldsymbol{\lambda})\in W(i,k+1)\subset V_{\alpha i}$$

$$E_j f_N(\boldsymbol{\lambda})\in W(i,k)\subset V_{\alpha i}$$

$$F_j f_N(\boldsymbol{\lambda})\in W(i,k)\subset V_{\alpha i}\qquad \text{for } j\neq i,i+1$$

that is to say, the space $V_{\alpha i}$ is invariant under the actions of $E_j, f_j, E_{i+1}$ and $F_i$ $(j\neq i,i+1)$.

Considering that all the vectors $f_N(\boldsymbol{\lambda})$ corresponding to all the lattice points $\boldsymbol{\lambda}$ in the hyperplane $\pi_i^\alpha$ satisfy

$$[\lambda_{i+1}-\lambda_i]=[\alpha p]=0$$

we have

$$E_i W(i,\alpha p)=0 \qquad F_{i+1} W(i,\alpha p)=0$$

and

$$E_i W(i,k)\subset W(i,k-1)\subset V_{\alpha i}$$
$$F_{i+1} W(i,k)\subset W(i,k-1)\subset V_{\alpha i} \qquad k=\alpha p+1,\alpha p+2,\ldots$$

namely, the subspace $V_{\alpha i}$ is also invariant under the actions of $E_i$ and $F_{i+1}$, and the theorem is proved.

According to theorem 3, there are many invariant subspaces $V_{\alpha i}$ corresponding to different hyperplanes $\Pi_i^\alpha$ for different $i$s and $\alpha$s. Like the analysis of $SL_q(2)$, the discussion of the reducibility of representation (19) results from the situations of the cross $V_{\alpha i}\cap V_{\alpha' i'}$ $(\alpha, i\neq\alpha', i')$. In the following section, we will use $SL_q(3)$ as an example to discuss this problem in detail.

## 6. Representations of $SL_q(3)$

When $p=3$, from (19), we obtain a representation of $SL_q(3)$:

$$\begin{aligned}
E_1 f_N(\lambda_1,\lambda_2)&=[\lambda_2-\lambda_1]f_N(\lambda_1+1,\lambda_2)\\
E_2 f_N(\lambda_1,\lambda_2)&=[N-\lambda_2]f_N(\lambda_1,\lambda_2+1)\\
F_1 f_N(\lambda_1,\lambda_2)&=[\lambda_1]f_N(\lambda_1-1,\lambda_2)\\
F_2 f_N(\lambda_1,\lambda_2)&=[\lambda_2-\lambda_1]f_N(\lambda_1,\lambda_2-1)\\
H_1 f_N(\lambda_1,\lambda_2)&=(2\lambda_1-\lambda_2)f_N(\lambda_1,\lambda_2)\\
H_2 f_N(\lambda_1,\lambda_2)&=(2\lambda_2-\lambda_1-N)f_N(\lambda_1,\lambda_2)
\end{aligned} \tag{22}$$

where $\lambda_2=0,1,2,\ldots,N$; $\lambda_1=0,1,2,\ldots,\lambda_2$ for a given $\lambda_2$. This representation is irreducible for the generic case.

In order to analyse the reducibility and decomposition of this representation when $q$ is a root of unity, we introduce the following 2D lattice diagram (figure 3) to describe

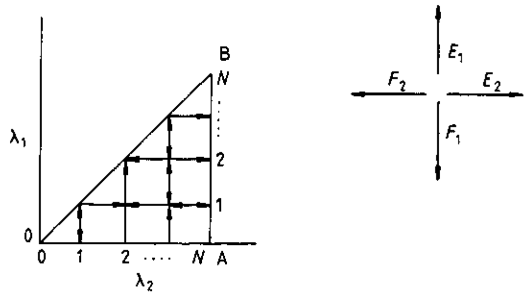


**Figure 3.** Diagram for the representation space $V_3^N$ and the actions of representation (22).

this representation. Here, each lattice point in $\triangle$OAB denotes a weight vector $f_N(\boldsymbol{\lambda})$; the upward, downward, right and left arrows denote the actions of $E_1$, $F_1$, $E_2$ and $F_2$ respectively.

The fact that $[kp]=0$ for $k\in\mathbb{Z}_+$ defines three character lines:

$$l_1:\quad \lambda_2-\lambda_1=\alpha p$$

$$l_2:\quad N-\lambda_2=\beta p$$

$$l_3:\quad \lambda_1=\gamma p \qquad \alpha,\beta,\nu\in\mathbb{Z}_+$$

which depict the reducibility of the representation (22). The three lines cut out of $V_3^N$: $\{f_N(\lambda_1,\lambda_2)\}$ three kinds of invariant subspaces,

$$V_\alpha(3):\quad \{f_N(\lambda_1,\lambda_2)\,|\,\lambda_2-\lambda_1\geqslant\alpha p\}$$

$$U_\beta(3):\quad \{f_N(\lambda_1,\lambda_2)\,|\,N-\lambda_2\geqslant\beta p\}$$

$$W_\gamma(3):\quad \{f_N(\lambda_1,\lambda_2)\,|\,\lambda_1\geqslant\gamma p\}$$

with the singular vectors $f_N(\lambda_1,\lambda_1+\alpha p)$, $f_N(\lambda_1,N-\beta p)$ and $f_N(\gamma p,\lambda_2)$ respectively. These vectors satisfy

$$E_1f_N(\lambda_1,\lambda_1+\alpha p)=F_2f_N(\lambda_1,\lambda_1+\alpha p)=0$$

$$E_2f_N(\lambda_1,N-\beta p)=0$$

$$F_2f_N(\gamma p,\lambda_2)=0.$$

The bases for these invariant subspaces $V_\alpha(3)$, $U_\beta(3)$ and $W_\gamma(3)$ respectively correspond to the lattice points in the shadowed domains of figures 4($a$–$c$).

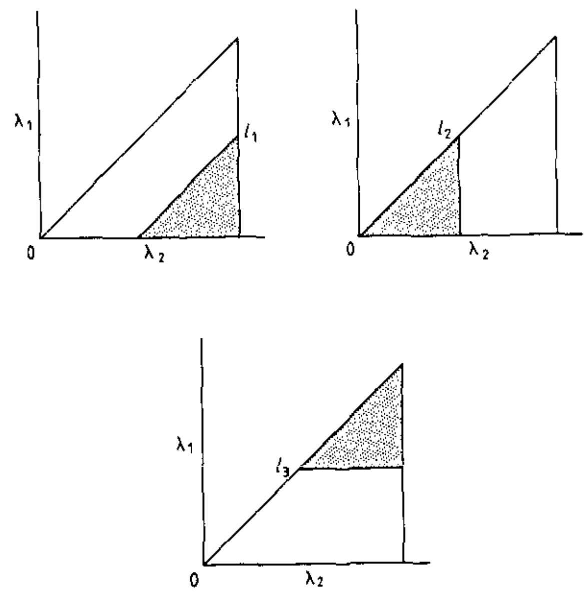


**Figure 4.** Diagrams for three types of invariant subspaces.

Considering that a cross of any two of these invariant subspaces is still invariant, we can obtain some lower-dimensional representations subduced by (22) on the following invariant subspaces:

$$Q_1 = V_\alpha(3) \cap U_\beta(3) \cap W_\gamma(3)$$

$$Q_2 = V_\alpha(3) \cap U_\beta(3)$$

$$Q_3 = U_\beta(3) \cap W_\gamma(3)$$

$$Q_4 = W_\gamma(3) \cap V_\alpha(3).$$

There are various situations of reducibility of spaces that are represented in figures 5($a$–$f$). Here, the shadowed domains correspond to invariant subspaces resulting from the crosses of original invariant subspaces.

Now, we calculate two representations of $\mathrm{SL}_q(3)$ from (22). When $p=3$ and $N=4$, we have a 15D indecomposable representation:

$$E_1 = E_{6,2} + E_{9,5} + E_{13,11} + E_{15,14} + [2](E_{7,3} + E_{11,8} + E_{14,12}) + E_{10,7}$$

$$F_1 = E_{2,6} + E_{3,7} + E_{5,9} + E_{4,8} + [2](E_{7,10} + E_{8,11} + E_{9,12}) + E_{14,15}$$

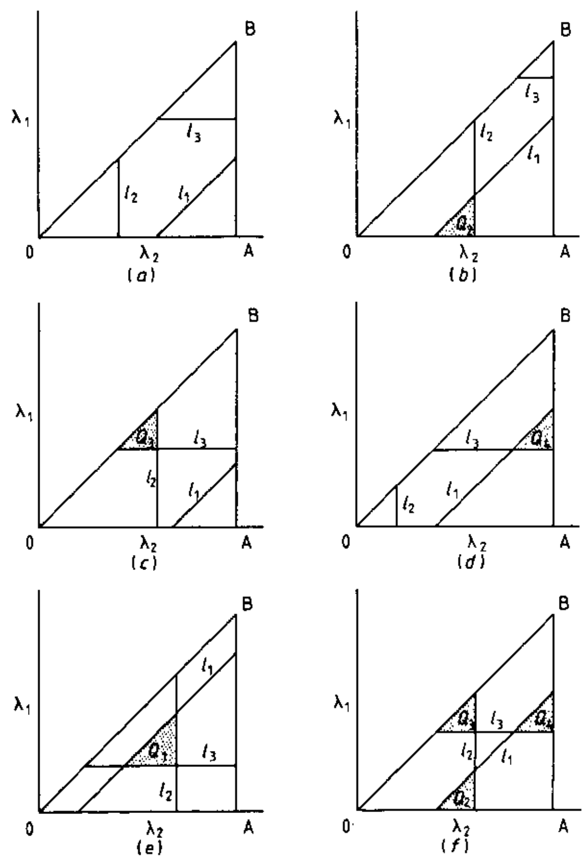


**Figure 5.** The invariant subspaces of $\mathrm{SL}_q(3)$: ($a$) $Q_1 = Q_2 = Q_3 = Q_4 = \{0\}$; ($b$) $Q_1 = Q_3 = Q_4 = \{0\}$, $Q_2 \neq \{0\}$; ($c$) $Q_1 = Q_2 = Q_4 = \{0\}$, $Q_3 \neq \{0\}$; ($d$) $Q_1 = Q_2 = Q_3 = \{0\}$, $Q_4 \neq \{0\}$; ($e$) $Q_2 = Q_3 = Q_4 = \{0\}$, $Q_1 \neq \{0\}$; ($f$) $Q_1 = \{0\}$, $Q_2$, $Q_3$, $Q_4 \neq \{0\}$.

$$
\begin{aligned}
E_2 &= E_{2,1}+E_{9,8}+E_{12,11}+E_{14,13}+[2](E_{3,4}+E_{8,7}+E_{11,10})+E_{5,4} \\
F_2 &= E_{1,2}+E_{4,5}+E_{6,7}+E_{10,11}+[2](E_{2,3}+E_{7,8}+E_{11,12})+E_{13,14} \\
H_1 &= -E_{2,2}-2E_{3,3}-3E_{4,4}-4E_{5,5}+E_{6,6}-E_{8,8}-2E_{9,9}+2E_{10,10}+E_{11,11} \\
&\quad +3E_{13,13}+2E_{14,14}+E_{15,15} \\
H_2 &= -4E_{1,1}-2E_{2,2}+2E_{4,4}+4E_{5,5}-3E_{6,6}-E_{7,7}+E_{8,8}+3E_{9,9}-2E_{10,10} \\
&\quad +2E_{12,12}-E_{13,13}+E_{14,14}.
\end{aligned}
\tag{23}
$$

From its representation diagram (figure 6), we observe that there exist three invariant subspaces

$$
\begin{aligned}
&S_1(3): \quad \{f_4(0,0), f_4(0,1), f_4(1,1)\} \\
&S_2(3): \quad \{f_4(0,3), f_4(0,4), f_4(1,4)\} \\
&S_3(3): \quad \{f_4(3,3), f_4(3,4), f_4(4,4)\}
\end{aligned}
$$

on which the representation (23) gives a 3D irreducible subrepresentation.

When $p=3$ and $N=5$, we obtain a 21D indecomposable representation:

$$
\begin{aligned}
E_1 &= E_{7,2}+E_{10,5}+E_{12,8}+E_{15,11}+E_{16,13}+E_{19,17}+E_{21,20} \\
&\quad +[2](E_{8,3}+E_{11,6}+E_{13,9}+E_{17,14}+E_{20,18}) \\
F_1 &= E_{2,7}+E_{3,8}+E_{4,9}+E_{5,10}+E_{6,11}+E_{17,19}+E_{18,20} \\
&\quad +[2](E_{8,12}+E_{9,13}+E_{10,14}+E_{11,15}+E_{20,21}) \\
E_2 &= E_{3,2}+E_{6,5}+E_{8,7}+E_{11,10}+E_{15,14}+E_{18,17}+E_{20,19} \\
&\quad +[2](E_{2,1}+E_{5,4}+E_{10,9}+E_{14,13}+E_{17,16}) \\
F_2 &= E_{1,2}+E_{4,5}+E_{7,8}+E_{10,11}+E_{12,13}+E_{16,17}+E_{19,20} \\
&\quad +[2](E_{2,3}+E_{5,6}+E_{8,9}+E_{13,14}+E_{17,18}) \\
H_1 &= -E_{2,2}-2E_{3,3}-3E_{4,4}-4E_{5,5}-5E_{6,6}+E_{7,7}-E_{9,9}-2E_{10,10}-3E_{11,11}+2E_{12,12} \\
&\quad +E_{13,13}-E_{15,15}+3E_{16,16}+E_{18,18}+4E_{19,19}+3E_{20,20}+5E_{21,21} \\
H_2 &= -5E_{1,1}-3E_{2,2}-E_{3,3}+E_{4,4}+3E_{5,5}+5E_{6,6}-4E_{7,7}-2E_{8,8}+E_{9,9}+2E_{10,10}+4E_{11,11} \\
&\quad -3E_{12,12}-E_{13,13}+E_{14,14}+3E_{15,15}-2E_{16,16}+2E_{18,18}-E_{19,19}+E_{20,20}.
\end{aligned}
\tag{24}
$$

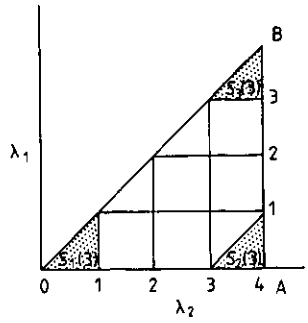


**Figure 6.** 15D indecomposable representation.

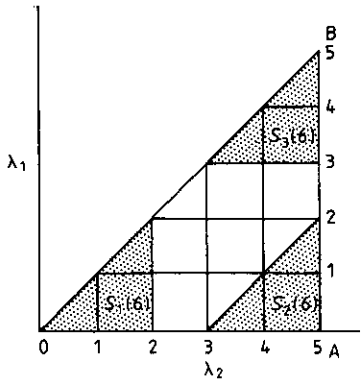


**Figure 7.** 21D indecomposable representation.

From its representation diagram (figure 7) we observe that there are three 6D invariant subspaces

$$S_1(6): \quad \{f_5(0,0), f_5(0,1), f_5(0,2), f_5(1,1), f_5(1,2), f_5(2,2)\}$$

$$S_2(6): \quad \{f_5(0,3), f_5(0,4), f_5(0,5), f_5(1,4), f_5(1,5), f_5(2,5)\}$$

$$S_3(6): \quad \{f_5(3,3), f_5(3,4), f_5(3,5), f_5(4,4), f_5(4,5), f_5(5,5)\}$$

on which the representation (24) subduces the 6D irreducible representations.

Finally, we point out that the problem will become very complicated when the Lusztig operators

$$\frac{E_1^p}{[p]!} \qquad \frac{E_2^p}{[p]!} \qquad \frac{F_1^p}{[p]!} \qquad \frac{F_2^p}{[p]!}$$

are introduced to extend $SL_q(3)$. Some details concerning this problem will be published elsewhere.

## Acknowledgments

The authors thank F Smirnov and L A Takhtajian for their stimulating discussions.

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